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SIMULTANEOUS HEAT AND MASS TRANSFER BETWEEN GAS AND LIQUID PHASES

I.—ANALYSIS OF UNSTEADY STATE TRANSFER

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Abstract—Unsteady-state simultaneous heat and mass transfer between air and aqueous solution are investigated to predict the liquid-phase heat- and mass-transfer coefficients. Expressions have been derived in rectangular, cylindrical and spherical co-ordinates, by combination of the film theory and the penetration theory. The effect of initial condition on the transfer coefficients is emphasized in these systems.

NOMENCLATURE

a_{ij}, boundary condition parameters as defined by equation (14);
C, concentration [kg/m^3];
D, diffusion coefficient [m^2/h];
h, heat-transfer coefficient [kcal/m^2 h degC];
H, humidity [kg/kg];
k, mass-transfer coefficient [kg/m^2 h ΔH];
K, over-all mass-transfer coefficient [kg/m^2 h ΔH];
L, characteristic length [m];
m, constant in equation (5) [(kg/kg)/degC];
n, constant in equation (5) [(kg/kg)/(kg/m^3)];
N, mass flux [kg/m^2 h];
q, heat flux [kcal/m^2 h];
r, radial co-ordinate [m];
t, temperature [°C];
U, over-all heat-transfer coefficient [kcal/m^2 h degC];
x, rectangular co-ordinate [m];
α_L, $\lambda/\rho C_p$ = thermal diffusivity [m^2/h];
α, dimensionless ratio as defined by equation (35);
γ, latent heat of evaporation [kcal/kg];
ζ, dimensionless quantity as defined by equation (61);
η, dimensionless quantity as defined by equation (57);
θ, time [h];
λ, thermal conductivity in liquid-phase [kcal/m h degC]
ξ, x/L or r/R;
ϕ, dimensionless parameter as defined by equation (63);
Sh, Sherwood number, Lk/D;
Nu, Nusselt number, Lh/λ;

Overline

$\bar{}$, time-average value;

Subscripts

G, gas phase quantity;
i, value at interface;
L, liquid phase quantity;
m, average value;
0, initial or over-all value.

INTRODUCTION

In recent years, simultaneous heat and mass transfer in the air and aqueous solution systems, as well as in the air–water system, have become an important problem of chemical engineering concerning dehumidification with dehydrating solutions, such as lithium chloride solution and triethylene glycol. It must be emphasized that for such systems the vapor–liquid equilibria depend upon not only temperature but liquid composition, and that there exist mass-transfer resistances on both gas and liquid sides. These facts necessarily cause the effects that the heat-transfer rate depends upon the mass-transfer rate, which in turn depends upon the heat-transfer rate.

In the present work the unsteady-state

simultaneous heat and mass transfer between phases are investigated. Combining the film theory and the pentration theory, the concentration and temperature profiles are obtained as a function of the contact time of the interface, and then the prediction of the liquid-phase heat- and mass-transfer coefficients is discussed. Some modification of the development described here may be applied to the problems of simultaneous heat and mass transfer in gas absorption, distillation, and so on, one of which has recently been reported by Modine, Parrish and Toor [1].

BASIC EQUATIONS AND THEIR SOLUTIONS

First let us consider the simplified unidimensional transport model, as shown in Fig. 1.

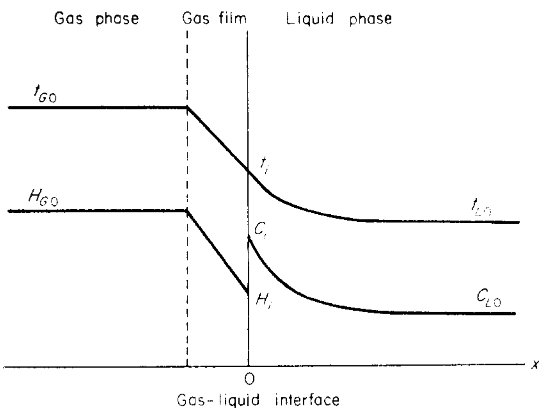

FIG. 1. Concentration and temperature profiles.

This problem is formulated by making the following assumptions [2, 3]:

(1) The film theory is applicable to the gas phase transfer. The transfer coefficients k_G and h_G, the bulk concentration H_{G0}, and the bulk temperature t_{G0} are constant.
(2) The penetration theory is applicable to the liquid phase transfer.
(3) The two phases are in equilibrium at the interface. In addition, the gas–liquid equilibrium relationship is linear.
(4) The sensible heat effects of mass transfer are ignored in both phases.
(5) The physical properties remain constant.
(6) The surface renewal effect is neglected.

Under these assumptions, the diffusion equation is

$$\frac{\partial C}{\partial \theta} = D_L \frac{\partial^2 C}{\partial x^2} \tag{1}$$

with the initial and boundary conditions at

$$\theta = 0, \quad C = C_{L0}, \tag{2a}$$

at

$$x = 0, \quad N_i = -D_L \left.\frac{\partial C}{\partial x}\right|_{x=0} = k_G (H_{G0} - H_i), \tag{2b}$$

at

$$x = \infty, \quad C = \text{finite}. \tag{2c}$$

The energy equation is

$$\frac{\partial t}{\partial \theta} = a_L \cdot \frac{\partial^2 t}{\partial x^2}, \tag{3}$$

with the initial and boundary conditions at

$$\theta = 0, \quad t = t_{L0}, \tag{4a}$$

at

$$x = 0, \quad q_i = -\lambda \left.\frac{\partial t}{\partial x}\right|_{x=0} = h_G (t_{G0} - t_i) + \gamma \cdot k_G (H_{G0} - H_i), \tag{4b}$$

at

$$x = \infty, \quad t = \text{finite}. \tag{4c}$$

The equilibrium relationship is given by

$$H_i = m \cdot t_i + n \cdot C_i + b. \tag{5}$$

The gas concentration C_{G0} corresponding to the bulk gas state (H_{G0}, t_{G0}) is defined by

$$H_{G0} = m \cdot t_{G0} + n \cdot C_{G0} + b. \tag{6}$$

It is convenient to define the following dimensionless quantities:

$$\xi = x/L, \tag{7}$$

$$y(\xi, \theta) = \frac{C(x, \theta) - C_{L0}}{C_{G0} - C_{L0}}, \tag{8}$$

$$z(\xi, \theta) = \frac{t(x, \theta) - t_{L0}}{t_{G0} - t_{L0}}. \tag{9}$$

The set of equations may now be expressed as the single matrix equation

$$\frac{\partial \mathbf{y}}{\partial \theta} = K \cdot \frac{\partial^2 \mathbf{y}}{\partial \xi^2}, \tag{10}$$

with the initial and boundary conditions

at

$$\theta = 0, \quad \mathbf{y} = 0, \tag{11a}$$

at

$$\xi = 0, \quad -\frac{\partial \mathbf{y}}{\partial \xi}\bigg|_{\xi=0} = \begin{bmatrix} \dfrac{N_i}{(D_L/L)(C_{G0} - C_{L0})} \\ \dfrac{q_i}{(\lambda/L)(t_{G0} - t_{L0})} \end{bmatrix} = \mathbf{b} - A \cdot \mathbf{y}_i, \tag{11b}$$

at

$$\xi = \infty, \quad \mathbf{y} = \text{finite}. \tag{11c}$$

where

$$\mathbf{y}(\xi, \theta) = \begin{bmatrix} y(\xi, \theta) \\ z(\xi, \theta) \end{bmatrix}, \quad \mathbf{y}_i = \begin{bmatrix} y(0, \theta) \\ z(0, \theta) \end{bmatrix}, \tag{12}$$

$$K = \begin{bmatrix} D_L/L^2 & 0 \\ 0 & \alpha_L/L^2 \end{bmatrix}, \quad A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \tag{13}$$

in which

$$\left.\begin{aligned} a_{11} &= \frac{k_G \cdot n}{D_L/L}, \quad a_{12} = \frac{k_G m(t_{G0} - t_{L0})}{(D_L/L)(C_{G0} - C_{L0})} \\ a_{21} &= \frac{\gamma \cdot k_G n (C_{G0} - C_{L0})}{(\lambda/L)(t_{G0} - t_{L0})}, \\ a_{22} &= \frac{h_G + \gamma k_G m}{\lambda/L} \\ b_1 &= a_{11} + a_{12}, \quad b_2 = a_{21} + a_{22}. \end{aligned}\right\} \tag{14}$$

This equation can be solved by means of Laplace transform (see Appendix) to obtain the following concentration and temperature profiles:

$$y(x, \theta) = \frac{\sqrt{(D_L)}/L}{\beta_1 - \beta_2}[c_{11} E_1(x, \theta, \beta_1) - c_{12} E_1(x, \theta, \beta_2)], \tag{15}$$

$$z(x, \theta) = \frac{\sqrt{(\alpha_L)}/L}{\beta_1 - \beta_2}[c_{21} \cdot E_2(x, \theta, \beta_1) - c_{22} E_2(x, \theta, \beta_2)], \tag{16}$$

where

$$E_1(x, \theta, \beta) = \text{erfc}\left[\frac{x}{2\sqrt{(D_L\theta)}}\right] - \exp\left[\frac{\beta x}{\sqrt{(D_L)}} + \beta^2\theta\right] \cdot \text{erfc}\left[\beta\sqrt{(\theta)} + \frac{x}{2\sqrt{(D_L\theta)}}\right], \tag{17}$$

$$E_2(x, \theta, \beta) = \text{erfc}\left[\frac{x}{2\sqrt{(\alpha_L\theta)}}\right] - \exp\left[\frac{\beta x}{\sqrt{(\alpha_L)}} + \beta^2\theta\right] \cdot \text{erfc}\left[\beta\sqrt{(\theta)} + \frac{x}{2\sqrt{(\alpha_L\theta)}}\right], \tag{18}$$

$$\left.\begin{aligned} \sigma &= |A| = a_{11} a_{22} - a_{12} a_{21} = \\ &= \frac{h_G k_G n}{D_L \lambda/L^2} \\ h_1 &= \frac{1}{L}[\sqrt{(D_L)} \cdot a_{11} + \sqrt{(\alpha_L)} \cdot a_{22}] \\ &= \frac{k_G n}{\sqrt{(D_L)}} + \frac{\sqrt{(\alpha_L)}(h_G + \gamma k_G m)}{\lambda} \\ h_2 &= \frac{\sqrt{(D_L \cdot \alpha_L)}}{L^2} \cdot \sigma = \frac{\sqrt{(\alpha_L)}}{\sqrt{(D_L)} \cdot \lambda} h_G k_G n \\ \beta_1 &= \frac{h_1 + \sqrt{(h_1^2 - 4h_2)}}{2}, \\ \beta_2 &= \frac{h_1 - \sqrt{(h_1^2 - 4h_2)}}{2} \\ \beta_1 - \beta_2 &= \sqrt{(h_1^2 - 4h_2)} \\ c_{11} &= b_1 - \sigma\sqrt{(\alpha_L)}/L\beta_1, \\ c_{12} &= b_1 - \sigma\sqrt{(\alpha_L)}/L\beta_2 \\ c_{21} &= b_2 - \sigma\sqrt{(D_L)}/L\beta_1, \\ c_{22} &= b_2 - \sigma\sqrt{(D_L)}/L\beta_2. \end{aligned}\right\} \tag{19}$$

The interfacial concentration and temperature may now be obtained by setting $x = 0$ in equations (15) and (16):

$$y_i(\theta) = \frac{C_i - C_{L0}}{C_{G0} - C_{L0}} = \frac{\sqrt{(D_L)}/L}{\beta_1 - \beta_2}[c_{11} E_0(\theta, \beta_1) - c_{12} E_0(\theta, \beta_2)], \tag{20}$$

$$z_i(\theta) = \frac{t_i - t_{L0}}{t_{G0} - t_{L0}} = \frac{\sqrt{(\alpha_L)}/L}{\beta_1 - \beta_2}[c_{21} E_0(\theta, \beta_1) - c_{22} E_0(\theta, \beta_2)], \tag{21}$$

where

$$E_0(\theta, \beta) = E_1(0, \theta, \beta) = E_2(0, \theta, \beta) = 1 - \exp(\beta^2\theta) \cdot \text{erfc}[\beta\sqrt{(\theta)}]. \tag{22}$$

The dimensionless mass- and heat- transfer rates are given by

$$\frac{N_i}{(D_L/L)(C_{G0} - C_{L0})} = \frac{1}{\beta_1 - \beta_2}\{c_{11}\,\beta_1 \exp(\beta_1^2\theta)\,\mathrm{erfc}\,[\beta_1\sqrt{(\theta)}] - c_{12}\,\beta_2 \exp(\beta_2^2\theta)\,.\,\mathrm{erfc}\,[\beta_2\sqrt{(\theta)}]\}, \quad (23)$$

$$\frac{q_i}{(\lambda/L)(t_{G0} - t_{L0})} = \frac{1}{\beta_1 - \beta_2}\{c_{21}\,\beta_1 \exp(\beta_1^2\theta)\,\mathrm{erfc}\,[\beta_1\sqrt{(\theta)}] - C_{22}\,\beta_2\,.\,\exp(\beta_2^2\theta)\,.\,\mathrm{erfc}\,[\beta_2\sqrt{(\theta)}]\}, \quad (24)$$

LIQUID-PHASE HEAT- AND MASS-TRANSFER COEFFICIENTS

For general considerations, we use the following expressions for the dimensionless mass- and heat-transfer rates:

$$F_1(\theta) = \frac{N_i(\theta)}{(D_L/L)(C_{G0} - C_{L0})}, \quad (25)$$

$$F_2(\theta) = \frac{q_i(\theta)}{(\lambda/L)(t_{G0} - t_{L0})}. \quad (26)$$

Over-all mass- and heat-transfer coefficients based on liquid-phase, K_{0L}, U are defined by

$$N_i(\theta) = K_{0L}(\theta)\,.\,(C_{G0} - C_{L0}), \quad (27)$$

$$q_i(\theta) = U(\theta)\,.\,(t_{G0} - t_{L0}). \quad (28)$$

Combining equations (25), (26) and (27), (28) respectively, we get

$$F_1(\theta) = \frac{K_{0L}(\theta)\,.\,L}{D_L} = Sh_L(\theta), \quad (29)$$

$$F_2(\theta) = \frac{U(\theta)\,.\,L}{\lambda} = Nu_L(\theta). \quad (30)$$

Individual liquid-phase mass- and heat-transfer coefficients k_L, h_L are defined by

$$N_i(\theta) = k_L(\theta)\,.\,(C_i - C_{L0}), \quad (31)$$

$$q_i(\theta) = h_L(\theta)\,.\,(t_i - t_{L0}). \quad (32)$$

Using equations (20), (21), (27) and (28)

$$k_L(\theta) = \frac{K_{0L}(\theta)}{y_i(\theta)} = \frac{(D_L/L)\,.\,F_1(\theta)}{y_i(\theta)}, \quad (33)$$

$$h_L(\theta) = \frac{U(\theta)}{z_i(\theta)} = \frac{(\lambda/L)\,.\,F_2(\theta)}{z_i(\theta)}. \quad (34)$$

Next the following dimensionless heat- and mass-transfer ratio will be introduced.

$$\alpha(\theta) \equiv \frac{q_i(\theta)}{\gamma\,.\,N_i(\theta)} = \frac{\lambda(t_{G0} - t_{L0})\,.\,F_2(\theta)}{\gamma\,.\,D_L(C_{G0} - C_{L0})\,.\,F_1(\theta)}. \quad (35)$$

Combining equations (2b), (4b), (5), (6), (27), (31) and (35), then gives

$$\frac{1}{K_{0L}(\theta)} = \frac{1}{n\,.\,k_G} + \frac{m\gamma(1 - \alpha)}{n\,.\,h_G} + \frac{1}{k_L(\theta)}. \quad (36)$$

A similar development for $U(\theta)$ gives

$$\frac{1}{U(\theta)} = \frac{\alpha - 1}{\alpha\,.\,h_G} + \frac{1}{h_L(\theta)}. \quad (37)$$

Finally, a time-average quantity for any function of time will be defined by the following expression:

$$\bar{F}(\theta) = \frac{1}{\theta}\int_0^\theta F(\theta')\,.\,d\theta'. \quad (38)$$

Then

$$\bar{N}_i(\theta) = \bar{K}_{0L}(\theta)\,.\,(C_{G0} - C_{L0}), \quad (39)$$

$$\bar{q}_i(\theta) = \bar{U}(\theta)\,.\,(t_{G0} - t_{L0}), \quad (40)$$

$$\bar{F}_1(\theta) = \frac{\bar{K}_{0L}(\theta)\,.\,L}{D_L} = \overline{Sh}_L(\theta), \quad (41)$$

$$\bar{F}_2(\theta) = \frac{\bar{U}(\theta)\,.\,L}{\lambda} = \overline{Nu}_L(\theta). \quad (42)$$

Defining the following average mass- and heat-transfer coefficients k_{Lm}, h_{Lm},

$$\bar{N}_i(\theta) = k_{Lm}(\theta)\,.\,(\bar{C}_i - C_{L0}), \quad (43)$$

$$\bar{q}_i(\theta) = h_{Lm}(\theta)\,.\,(\bar{t}_i - t_{L0}), \quad (44)$$

where

$$k_{Lm}(\theta) = \frac{\bar{K}_{0L}(\theta)}{\bar{y}_i(\theta)} = \frac{(D_L/L)\,\bar{F}_1(\theta)}{\bar{y}_i(\theta)}, \quad (45)$$

$$h_{Lm}(\theta) = \frac{\bar{U}(\theta)}{\bar{z}_i(\theta)} = \frac{(\lambda/L)\,\bar{F}_2(\theta)}{\bar{z}_i(\theta)}, \quad (46)$$

then, corresponding to equations (36) and (37), similar relations for $\bar{K}_{0L}(\theta)$ and $\bar{U}(\theta)$ are

$$\frac{1}{\bar{K}_{0L}(\theta)} = \frac{1}{n\,.\,k_G} + \frac{m\gamma\,(1-a_m)}{n\,.\,h_G} + \frac{1}{k_{Lm}(\theta)}, \tag{47}$$

$$\frac{1}{\bar{U}(\theta)} = \frac{a_m - 1}{a_m\,.\,h_G} + \frac{1}{h_{Lm}(\theta)}, \tag{48}$$

in which

$$a_m(\theta) = \frac{\bar{q}_i(\theta)}{\gamma\,.\,\bar{N}_i(\theta)} = \frac{\lambda\,(t_{G0}-t_{L0})\,.\,\bar{F}_2(\theta)}{D_L\,(C_{G0}-C_{L0})\,.\,\bar{F}_1(\theta)}. \tag{49}$$

The required time-average quantities are summarized as

$$\bar{F}_k(\theta) = \frac{1}{\beta_1-\beta_2}[c_{k1}\,\beta_1\,\bar{F}_0(\theta,\beta_1) - c_{k2}\,\beta_2\,\bar{F}_0(\theta,\beta_2)] \quad (k=1,2), \tag{50}$$

where

$$\left.\begin{aligned} \bar{F}_0(\theta,\beta) &= \frac{1}{\beta^2\theta}\left\{2\beta\sqrt{\left(\frac{\theta}{\pi}\right)} + \exp(\beta^2\theta)\,.\,\operatorname{erfc}[\beta\sqrt{(\theta)}] - 1\right\} \quad \ldots.\ (\beta \neq 0) \\ &= 1 \quad \ldots\ldots\ldots\ldots\ldots\ (\beta = 0) \end{aligned}\right\} \tag{51}$$

$$\bar{y}_i(\theta) = \frac{\bar{C}_i - C_{L0}}{C_{G0}-C_{L0}} = \frac{\sqrt{(D_L)}/L}{\beta_1-\beta_2}[c_{11}\,\bar{E}_0(\theta,\beta_1) - c_{12}\,\bar{E}_0(\theta,\beta_2)], \tag{52}$$

$$\bar{z}_i(\theta) = \frac{\bar{t}_i - t_{L0}}{t_{G0}-t_{L0}} = \frac{\sqrt{(a_L)}/L}{\beta_1-\beta_2}[c_{21}\,\bar{E}_0(\theta,\beta_1) - c_{12}\,\bar{E}_0(\theta,\beta_2)], \tag{53}$$

where

$$\bar{E}_0(\theta,\beta) = 1 - \bar{F}_0(\theta,\beta), \tag{54}$$

$$\frac{\bar{F}_2(\theta)}{\bar{F}_1(\theta)} = \frac{c_{21}\,\beta_1\,\bar{F}_0(\theta,\beta_1) - c_{22}\,\beta_2\,\bar{F}_0(\theta,\beta_2)}{c_{11}\,\beta_1\,\bar{F}_0(\theta,\beta_1) - c_{12}\,\beta_2\,\bar{F}_0(\theta,\beta_2)}. \tag{55}$$

NUMERICAL EXAMPLES

(Case I) Mass transfer only

For the mass transfer in the absence of heat transfer, equation (50) reduces to [4, 5]

$$\overline{Sh}_L(\theta) = \bar{F}_1(\theta) = \frac{\bar{N}_i(\theta)}{(D_L/L)\,(C_{G0}-C_{L0})} = a_{11}\,f(\eta), \tag{56}$$

where

$$\eta = \frac{k_G n}{\sqrt{(D_L)}}\,.\,\sqrt{(\theta)} \tag{57}$$

$$f(\eta) = \frac{2}{\sqrt{(\pi)}\,.\,\eta} + \frac{\exp\eta^2\,.\,\operatorname{erfc}\eta - 1}{\eta^2}. \tag{58}$$

Equation (56) is plotted in Fig. 2.

For the liquid-phase mass-transfer coefficient, using equations (45), (50) and (52),

$$\frac{k_{Lm}}{k_G\,.\,n} = \frac{f(\eta)}{1-f(\eta)}. \tag{59}$$

This result is shown in Fig. 3.

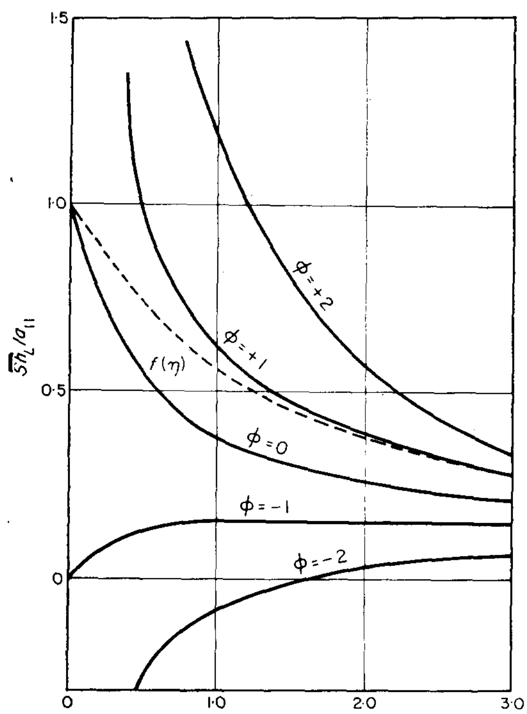

FIG. 2. Over-all mass-transfer coefficients.

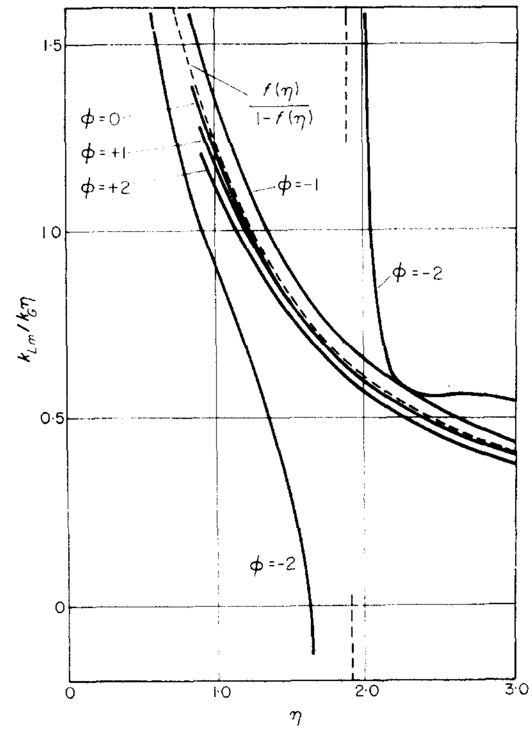

FIG. 3. Liquid-phase mass-transfer coefficients.

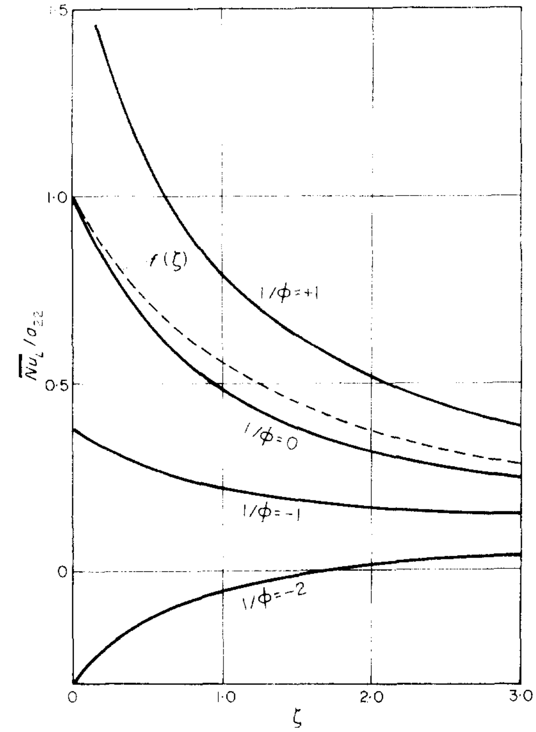

FIG. 4. Over-all heat-transfer coefficients.

(*Case II*) *The air–water system*

For the heat transfer with mass transfer in which $n = 0$, as in the air–water system, equation (50) does not hold. This leads to the following modification:

$$\overline{Nu}_L(\theta) = F_2(\theta) = \frac{\bar{q}_i(\theta)}{(\lambda/L)(t^*_{G0} - t_{L0})} = a_{22} f(\zeta) \tag{60}$$

where

$$\left.\begin{aligned} \zeta &= \frac{\sqrt{(a_L)}\,(h_G + \gamma k_G \,.\, m)}{\lambda} \sqrt{(\theta)} \\ t^*_{G0} &= \frac{h_G \,.\, t_{G0} + \gamma k_G (H_{G0} - b)}{h_G + \gamma k_G \,.\, m} \end{aligned}\right\} \tag{61}$$

$f(\zeta)$ is also defined by equation (58). Equation (60) is plotted in Fig. 4. For the liquid-phase heat-transfer coefficient, the following expression can be obtained:

$$\frac{h_{Lm}}{h_G + \gamma k_G \,.\, m} = \frac{f(\zeta)}{1 - f(\zeta)} \tag{62}$$

This result is shown in Fig. 5.

(*Case III*) *The air–aqueous solution system*

A numerical analysis for this system is made on the following assumptions:

$$\begin{aligned} m &= 0{\cdot}0009\ [(\text{kg/kg})/\text{degC}] \\ n &= 0{\cdot}04 \times 10^{-3}\ [(\text{kg/kg})/(\text{kg/m}^3)] \\ D_L &= 1{\cdot}5 \times 10^{-5}\ [\text{cm}^2/\text{s}] = 0{\cdot}54 \times 10^{-5}\ [\text{m}^2/\text{h}] \\ a_L &= 1{\cdot}5 \times 10^{-3}\ [\text{cm}^2/\text{s}] = 0{\cdot}54 \times 10^{-3}\ [\text{m}^2/\text{h}] \\ \gamma &= 595\ [\text{kcal/kg}] \\ \lambda &= 0{\cdot}51\ [\text{kcal/m h degC}] \\ h_G &= 30\ [\text{kcal/m}^2\ \text{h degC}] \\ k_G &= 120\ [\text{kg/m}^2\text{h (kg/kg)}]. \end{aligned}$$

For convenience, the following dimensionless quantity is defined.

$$\phi = \frac{a_{12}}{a_{11}} = \frac{m\,(t_{G0} - t_{L0})}{n \,.\, (C_{G0} - C_{L0})}. \tag{63}$$

Then, for mass transfer, equations (45), (50) and (52) give $\overline{Sh}_L/a_{11}$ and $k_{Lm}/k_G n$ as functions of η and ϕ. For comparison with the results of Case I, these numerical values are shown in

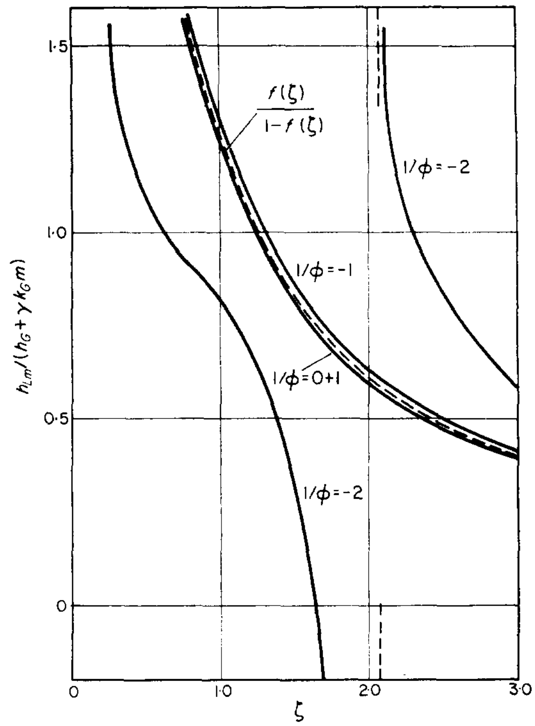

FIG. 5. Liquid-phase heat-transfer coefficients.

Figs. 2 and 3 respectively. Similarly, for heat transfer, $\overline{Nu}_L/a_{22}$ and $h_{Lm}/(h_G + \gamma k_G m)$ are obtained from equations (46), (50) and (53) as functions of ζ and ϕ. For comparison with the results of Case II, these results are also presented in Figs. 4 and 5 respectively.

In the limit as $\theta \to 0$ (thus η and $\zeta \to 0$),

$$\left.\begin{aligned} \overline{Sh}_L/a_{11} &= 1 + \phi \\ \overline{Nu}_L/a_{22} &= 1 + \frac{\gamma k_G \,.\, m}{h_G + \gamma k_G \,.\, m} \cdot \frac{1}{\phi} \\ k_{Lm}, h_{Lm} &\longrightarrow +\infty \end{aligned}\right\} . \quad (64)$$

It is to be noted that the results for Case III show a dependence on the initial condition parameter ϕ that is not found in the other cases. As can be seen from Figs. 2, 3, 4 and 5 this effect of ϕ is to a greater extent on the over-all transfer coefficients (See Figs. 2 and 4) than on the liquid-phase transfer coefficients (See Figs. 3 and 5), except for the case of ϕ or $1/\phi = -2$.

In this case, the direction of the mass or heat transfer will be changed in the course of transfer processes. This predicts the greatest effect of ϕ on the transfer coefficients. When $\bar{y}_i(\theta) = 0$, so that the liquid-phase concentration driving force $(\bar{C}_i - C_{L0}) = 0$, it is shown that the mass flux $\bar{N}_i(\theta)$ or $\bar{F}_1(\theta)$ does not vanish. It is the same with heat transfer.

RESULTS IN CURVILINEAR CO-ORDINATES

The previous results for the simplified unidimensional model will be applicable to the analysis of the data, for example, in wetted-wall columns. For a jet and a spray of liquid, however, the corresponding results may be required in curvilinear co-ordinates.

In these cases, the basic equation reduces to

$$\frac{\partial y}{\partial \theta} = K \,.\, \nabla^2 y \quad (65)$$

with the initial and boundary conditions

$$\text{at} \quad \theta = 0, \quad y = 0 \quad (66a)$$

$$\text{at} \quad \xi = 0, \quad y = \text{finite} \quad (66b)$$

$$\text{at} \quad \xi = 1,$$

$$\left.\frac{\partial y}{\partial \xi}\right|_{\xi=1} = \begin{bmatrix} \dfrac{N_i}{(D_L/R)(C_{G0} - C_{L0})} \\ \dfrac{q_i}{(\lambda/R)(t_{G0} - t_{L0})} \end{bmatrix} = \ell - A \,.\, y_i \quad (66c)$$

with

$$y = y(\xi, \theta), \quad y_i = y(1, \theta) \quad (67)$$

ξ, K, A, and ℓ are given by equations (7), (13) and (14) respectively in which $L = R$.

The Laplacian operator is obtained by

$$\nabla^2 = \begin{cases} \dfrac{1}{\xi} \cdot \dfrac{\partial}{\partial \xi}\left(\xi \dfrac{\partial}{\partial \xi}\right) \text{ for cylindrical co-ordinates} & (68a) \\ \dfrac{1}{\xi^2} \cdot \dfrac{\partial}{\partial \xi}\left(\xi^2 \dfrac{\partial}{\partial \xi}\right) \text{ for spherical co-ordinates} & (68b) \end{cases}$$

The desired solutions (see Appendix) are summarized in Tables 1 and 2.

Table 1. Results in cylindrical co-ordinates

$$y(\xi,\theta) = 1 - 2\sum_{k=1}^{\infty} A_{1k} J_0(\gamma_{1k}\xi)\exp(-\gamma_{1k}^2 D_L\theta/R^2) \tag{69}$$

$$z(\xi,\theta) = 1 - 2\sum_{l=1}^{\infty} A_{2l} J_0(\gamma_{2l}\xi)\exp(-\gamma_{2l}^2 \alpha_L\theta/R^2) \tag{70}$$

where

$$A_{1k} = \frac{\sigma\cdot J_0(\gamma_{2k}) - \gamma_{2k} b_1 J_1(\gamma_{2k})}{(\gamma_{2k}^2/D_L)\cdot\Phi_k} = \frac{b_1 J_0(\gamma_{1k})}{\gamma_{1k}^2[\{J_0(\gamma_{1k})\}^2 + \{J_1(\gamma_{1k})\}^2]} \tag{71}$$

$$A_{2l} = \frac{\sigma\cdot J_0(\gamma_{1l}) - \gamma_{1l} b_2 . J_1(\gamma_{1l})}{(\gamma_{1l}^2/\alpha_L)\cdot\Phi_l} = \frac{b_2 J_0(\gamma_{2l})}{\gamma_{2l}^2[\{J_0(\gamma_{2l})\}^2 + \{J_1(\gamma_{2l})\}^2]} \tag{72}$$

$(k, l = 1, 2, \ldots)$

$$\Phi_k = \left(\frac{\sigma\cdot\alpha_L}{\gamma_{1k}} - D_L\gamma_{1k}\right)J_1(\gamma_{1k})J_0(\gamma_{2k}) + \left(\frac{\sigma\cdot D_L}{\gamma_{2k}} - \alpha_L\gamma_{2k}\right)\cdot J_0(\gamma_{1k})J_1(\gamma_{2k}) + (D_L a_{11} + \alpha_L a_{22})J_0(\gamma_{1k})J_0(\gamma_{2k})$$
$$- \sqrt{(D_L\cdot\alpha_L)}(a_{11} + a_{22})J_1(\gamma_{1k})J_1(\gamma_{2k}) \tag{73}$$

in which $\gamma_k\,[=\gamma_{1k}\sqrt{(D_L)} = \gamma_{2k}\sqrt{(\alpha_L)}]$'s are the characteristic roots of the following equation (74).

$$0 = \Delta(\gamma) \equiv \sigma\cdot J_0(\gamma_1)J_0(\gamma_2) - a_{22}\gamma_1 J_1(\gamma_1)J_0(\gamma_2) - a_{11}\gamma_2 J_0(\gamma_1)J_1(\gamma_2) + \gamma_1\gamma_2 J_1(\gamma_1)J_1(\gamma_2). \tag{74}$$

In addition,

$$\frac{N_i(\theta)}{(D_L/R)(C_{G0} - C_{L0})} = F_1(\theta) = 2\sum_{k=1}^{\infty} A_{1k}\gamma_{1k} J_1(\gamma_{1k})\cdot\exp(-\gamma_{1k}^2 D_L\theta/R^2) \tag{75}$$

$$\frac{q_i(\theta)}{(\lambda/R)(t_{G0} - t_{L0})} = F_2(\theta) = 2\sum_{l=1}^{\infty} A_{2l}\gamma_{2l} J_1(\gamma_{2l})\cdot\exp(-\gamma_{2l}^2 \alpha_L\theta/R^2) \tag{76}$$

Table 2. Results in spherical co-ordinates

$$y(\xi,\theta) = 1 + 2\sum_{k=1}^{\infty} B_{1k}\frac{\sin(\gamma_{1k}\xi)}{\xi}\cdot\exp(-\gamma_{1k}^2 D_L\theta/R^2) \tag{77}$$

$$z(\xi,\theta) = 1 + 2\sum_{l=1}^{\infty} B_{2l}\frac{\sin(\gamma_{2l}\xi)}{\xi}\cdot\exp(-\gamma_{2l}^2 \alpha_L\theta/R^2) \tag{78}$$

where

$$B_{1k} = \frac{\sigma_1\cdot\sin\gamma_{2k} + \gamma_{2k}\cdot b_1\cdot\cos\gamma_{2k}}{(\gamma_{2k}^2/D_L)\cdot\Phi_k} = \frac{b_1}{\gamma_{1k}^2\left[\dfrac{\cos\gamma_{1k}}{\gamma_{1k}} - \dfrac{1}{\sin\gamma_{1k}}\right]} \tag{79}$$

$$B_{2l} = \frac{\sigma_2\cdot\sin\gamma_{1l} + \gamma_{1l}\cdot b_2\cdot\cos\gamma_{1l}}{(\gamma_{1l}^2/\alpha_L)\cdot\Phi_l} = \frac{b_2}{\gamma_{2l}^2\left[\dfrac{\cos\gamma_{2l}}{\gamma_{2l}} - \dfrac{1}{\sin\gamma_{2l}}\right]} \tag{80}$$

$(k, l = 1, 2, \ldots)$

$$\Phi_k = \left[\frac{\sigma' + (a_{22} - 1)}{\gamma_{1k}}\alpha_L - D_L\gamma_{1k}\right]\cos\gamma_{1k}\cdot\sin\gamma_{2k} + \left[\frac{\sigma' + (a_{11} - 1)}{\gamma_{2k}}D_L - \alpha_L\gamma_{2k}\right]\sin\gamma_{1k}\cdot\cos\gamma_{2k}$$
$$- [D_L(a_{11} - 1) + \alpha_L(a_{22} - 1)]\sin\gamma_{1k}\cdot\sin\gamma_{2k} + \sqrt{(D_L\cdot\alpha_L)}\cdot(a_{11} + a_{22})\cos\gamma_{1k}\cdot\cos\gamma_{2k}, \tag{81}$$

Table 2—continued

in which

$$\left.\begin{aligned}\sigma' &= (a_{11} - 1)(a_{22} - 1) - a_{12}a_{21}\\ \sigma_1 &= b_1(a_{22} - 1) - b_2 a_{12}\\ \sigma_2 &= b_2(a_{11} - 1) - b_1 \cdot a_{21}\end{aligned}\right\}. \tag{82}$$

$\gamma_k[= \gamma_{1k}\sqrt{(D_L)} = \gamma_{2k}\sqrt{(\alpha_L)}]$'s are the characteristic roots of the following equation (83).

$$0 = -\Delta(\gamma) \equiv \sigma' \cdot \sin\gamma_1 \cdot \sin\gamma_2 + (a_{22} - 1)\gamma_1\cos\gamma_1 \cdot \sin\gamma_2 + (a_{11} - 1)\gamma_2 \cdot \sin\gamma_1 \cdot \cos\gamma_2 + \gamma_1\gamma_2\cos\gamma_1 \cdot \cos\gamma_2. \tag{83}$$

In addition,

$$\frac{N_i(\theta)}{(D_L/R)(C_{G0} - C_{L0})} = F_1(\theta) = 2\sum_{k=0}^{\infty} B_{1k}(\gamma_{1k}\cos\gamma_{1k} - \sin\gamma_{1k}) \cdot \exp(-\gamma_{1k}^2 D_L\theta/R^2), \tag{84}$$

$$\frac{q_i(\theta)}{(\lambda/R)(t_{G0} - t_{L0})} = F_2(\theta) = 2\sum_{l=1}^{\infty} B_{2l}(\gamma_{2l}\cos\gamma_{2l} - \sin\gamma_{2l}) \cdot \exp(-\gamma_{2l}^2 \alpha_L\theta/R^2). \tag{85}$$

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APPENDIX

Notes on Derivation

For the present purpose, it is convenient to use the Laplace-transform method and matrix notation [6–9].

(1) *Rectangular co-ordinates*

The Laplace transform of $\mathscr{y}(\xi, \theta)$ with respect to θ is defined by

$$\left.\begin{aligned}\mathscr{L}[\mathscr{y}] &= \mathscr{Y}(\xi, s) = \int_0^\infty \mathscr{y}(\xi, \theta)\,.\,e^{-s\theta}\,d\theta\\ &= \begin{bmatrix}\mathscr{L}[y]\\ \mathscr{L}[z]\end{bmatrix} = \begin{bmatrix}Y(\xi, s)\\ Z(\xi, s)\end{bmatrix}\end{aligned}\right\}. \tag{A.1}$$

Then, the Laplace transform of the foregoing basic equation and its boundary conditions (10), (11) reduces to the following ordinary differential equation:

$$\frac{d^2\mathscr{Y}}{d\xi^2} = Q^2\,.\,\mathscr{Y}, \tag{A.2}$$

with the boundary conditions

at

$$\xi = 0, \quad -\left.\frac{d\mathscr{Y}}{d\xi}\right|_{\xi=0} = \frac{\mathscr{b}}{s} - A\,.\,\mathscr{Y}_i \tag{A.3a}$$

at

$$\xi = \infty, \quad \mathscr{Y} = \text{finite}, \tag{A.3b}$$

where

$$Q = \begin{bmatrix}L\sqrt{(s/D_L)} & 0\\ 0 & L\sqrt{(s/\alpha_L)}\end{bmatrix}.$$

The solution of equation (A.2), (A.3a) and (A.3b) is readily found to be

$$\mathscr{Y}(\xi, s) = \exp(-Q\xi)\,.\,\mathscr{C} = \begin{Bmatrix}C_1\,.\,\exp[-\xi L\sqrt{(s/D_L)}]\\ C_2\,.\,\exp[-\xi L\sqrt{(s/\alpha_L)}]\end{Bmatrix} \tag{A.4}$$

where

$$C_1 = \frac{[\sqrt{(D_L)}/L^2][b_1 L\sqrt{(s)} + \sigma\sqrt{(\alpha_L)}]}{s[\sqrt{(s)} + \beta_1][\sqrt{(s)} + \beta_2]}$$

$$C_2 = \frac{[\sqrt{(\alpha_L)}/L^2][b_2 L\sqrt{(s)} + \sigma\sqrt{(D_L)}]}{s[\sqrt{(s)} + \beta_1][\sqrt{(s)} + \beta_2]}.$$

The inverse transform $y(\xi, \theta) = \mathscr{L}^{-1}[\mathscr{Y}(\xi, s)]$ may be obtained by use of the transform relation

$$\mathscr{L}^{-1}\left\{\frac{a \cdot \exp[-k\sqrt{(s)}]}{s[\sqrt{(s)} + a]}\right\} = \text{erfc}\,[k/2\sqrt{(\theta)}]$$

$$- \exp(ak + a^2\theta) \cdot \text{erfc}\left[a\sqrt{(\theta)} + \frac{k}{2\sqrt{(\theta)}}\right].$$

The final results are given by equation (15) and (16).

(2) *Cylindrical co-ordinates*

The Laplace transform of equations (65) and (66) reduces to

$$\frac{d^2\mathscr{Y}}{d\xi^2} + \frac{1}{\xi} \cdot \frac{d\mathscr{Y}}{d\xi} - Q^2\mathscr{Y} = 0 \qquad \text{(A.5)}$$

with the boundary conditions

$$\text{at} \qquad \xi = 0, \quad \mathscr{Y} = \text{finite} \qquad \text{(A.6a)}$$

$$\text{at} \quad \xi = 1, \quad \left.\frac{d\mathscr{Y}}{d\xi}\right|_{\xi=1} = \frac{\ell}{s} - A \cdot \mathscr{Y}_i. \qquad \text{(A.6b)}$$

The transform solution is obtained in terms of Bessel functions

$$\mathscr{Y}(\xi, s) = J_0(i \cdot Q\xi) \cdot \mathscr{C} = \begin{Bmatrix} C_1 J_0[i\xi R\sqrt{(s/D_L)}] \\ C_2 J_0[i\xi R\sqrt{(s/\alpha_L)}] \end{Bmatrix}, \qquad \text{(A.7)}$$

where

$$C_1 = \frac{\sigma \cdot J_0(a_2) + a_2 b_1 J_0'(a_2)}{s \cdot \Delta(s)},$$

$$C_2 = \frac{\sigma \cdot J_0(a_1) + a_1 b_2 J_0'(a_1)}{s \cdot \Delta(s)},$$

with

$$a_1 = iR\sqrt{(s/D_L)}, \qquad a_2 = iR\sqrt{(s/\alpha_L)},$$

$$\Delta(s) \equiv \sigma J_0(a_1) J_0(a_2) + a_1 a_{22} J_0'(a_1) J_0(a_2) + a_2 a_{11} J_0(a_1) J_0'(a_2) + a_1 a_2 J_0'(a_1) J_0'(a_2).$$

The inversion of equation (A.7) may be accomplished by means of the Heaviside partial fractions expansion theorem (or the method of residues) and several properties of Bessel functions. The final results are shown in Table 1.

(3) *Spherical co-ordinates*

If we make the substitution

$$\mathscr{U}(\xi, \theta) = \xi \cdot y(\xi, \theta) \qquad \text{(A.8)}$$

the equations (65) and (66) are

$$\frac{\partial \mathscr{U}}{\partial \theta} = K \frac{\partial^2 \mathscr{U}}{\partial \xi^2} \qquad \text{(A.9)}$$

$$\text{at} \quad \xi = 0, \quad \mathscr{U} = 0 \qquad \text{(A.10a)}$$

$$\text{at} \quad \xi = 1, \quad \left.\frac{\partial \mathscr{U}}{\partial \xi}\right|_{\xi=1} = \ell - (A - E)\,\mathscr{U}_i \quad \text{(A.10b)}$$

where E = unit matrix.

This problem can now be solved by the methods mentioned above.

Résumé—On recherche le transport de chaleur et de masse simultané en régime non permanent entre l'air et une solution aqueuse pour prédire les coefficients de transport de chaleur et de masse. On a obtenu des expressions en coordonnées rectangulaires cylindriques et sphériques, en combinant la théorie du "film" et la théorie de la pénétration. L'effet de la condition initiale sur les coefficients de transport est accentué dans ces systèmes.

Zusammenfassung—Der instationäre, simultan ablaufende Wärme- und Stofftransport zwischen Luft und wässriger Lösung wurde untersucht, um Wärme- und Stoffübergangskoeffizienten der flüssigen Phase zu bestimmen. Durch Kombination der Film und der Eindringtheorie liessen sich Gleichungen ableiten sowohl für rechtwinklige Koordinaten, als auch für Zylinder- und Kugelkoordinaten. Der Einfluss der Anfangsbedingungen auf die Übergangskoeffizienten wird in den Systemen deutlich.

Аннотация—Исследуется нестационарный процесс тепло-и массообмена между воздухом и водным раствором различных веществ с целью определения коэффициентов тепло-и массопереноса для жидкой и газообразной фаз. Путем совместного применения пленочной теории и теории проникновения получены соответствующие выражения в прямоугольных, цилиндрических и сферических координатах. Подчеркивается влияние начальных условий на коэффициенты переноса в этих системах.